

The splitting problem for complex homogeneous supermanifolds ¹

E.G. Vishnyakova

Abstract

It is a classical result that any complex analytic Lie supergroup \mathcal{G} is split [5], that is its structure sheaf is isomorphic to the structure sheaf of a certain vector bundle. However, there do exist non-split complex analytic homogeneous supermanifolds.

We study the question how to find out whether a complex analytic homogeneous supermanifold is split or non-split. Our main result is a description of left invariant gradings on a complex analytic homogeneous supermanifold \mathcal{G}/\mathcal{H} in the terms of \mathcal{H} -invariants. As a corollary to our investigations we get some simple sufficient conditions for a complex analytic homogeneous supermanifold to be split in terms of Lie algebras.

1 Introduction

A supermanifold is called split if its structure sheaf is isomorphic to the exterior power of a certain vector bundle. By Batchelor's Theorem any real supermanifold is non-canonically split. However, this is false in the complex analytic case. The property of a supermanifold to be split is very important for several reasons. For instance, in [2] it was shown that the moduli space of super Riemann surfaces is not projected (and in particular is not split) for genus $g \geq 5$. The physical meaning of this result is that [2]: "certain approaches to superstring perturbation theory that are very powerful in low orders have no close analog in higher orders". Another problem, when the property of a supermanifold to be split is very important, is the calculation of the cohomology group with values in a vector bundle over a supermanifold. In the split case we may use the well understood tools of complex analytic geometry. In the general case, several methods were suggested by Onishchik's school: spectral sequences, see e.g. [12]. All these methods connect the cohomology group with values in a vector bundle with the cohomology group with values in the corresponding split vector bundle.

¹Supported by Max Planck Institute for Mathematics Bonn and AFR-grant, University of Luxembourg.

How do we determine whether a complex analytic supermanifold is split or non-split? Let me describe here some results in this direction that were obtained by Green, Koszul, Onishchik and Serov. In [3] Green described a moduli space with a marked point such that any non-marked point corresponds to a non-split supermanifold while the marked point corresponds to a split one. His idea was used for instance in [2]. The calculation of the Green moduli space is a difficult problem itself, and in many cases the method is difficult to apply. Furthermore, Onishchik and Serov [9, 10, 11] considered grading derivations, which correspond to \mathbb{Z} -gradings of the structure sheaf of a supermanifold. For example, it was shown that almost all supergrassmannians do not possess such derivations, i.e. their structure sheaves do not possess any \mathbb{Z} -gradings. Hence, in particular, they are non-split. The idea of grading derivations was independently used by Koszul. In [4] the following statement was proved: if the tangent bundle of a supermanifold \mathcal{M} possesses a (holomorphic) connection then \mathcal{M} is split. (Koszul's proof works in real and complex analytic cases.) In fact, it was shown that we can assign a grading derivation to any supermanifold with a connection and that this grading derivation is induced by a \mathbb{Z} -grading of a vector bundle.

Assume that a complex analytic supermanifold $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ is split. By definition this means that its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$, where \mathcal{E} is a locally free sheaf on the complex analytic manifold \mathcal{M}_0 . The sheaf $\bigwedge \mathcal{E}$ is naturally \mathbb{Z} -graded and the isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ induces the \mathbb{Z} -grading in $\mathcal{O}_{\mathcal{M}}$. We call such gradings *split*. The main result of our paper is a description of those left invariant split gradings on a homogeneous superspace \mathcal{G}/\mathcal{H} which are compatible with split gradings on \mathcal{G} . We also give sufficient conditions for pairs $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \text{Lie } \mathcal{G}$ and $\mathfrak{h} = \text{Lie } \mathcal{H}$, such that \mathcal{G}/\mathcal{H} is split.

Acknowledgment. The author is grateful to A. Onishchik, V. Serganova, P. Teichner and R. Donagi for their attention to this work and anonymous referee for useful comments.

2 Complex analytic supermanifolds. Main definitions.

We will use the word "supermanifold" in the sense of Berezin and Leites, see [1], [7] and [8] for details. Throughout, we will be interested in the complex analytic version of the theory. Recall that a *complex analytic superdomain*

of dimension $n|m$ is a \mathbb{Z}_2 -graded ringed space

$$\mathcal{U} = \left(U, \mathcal{F}_U \otimes \bigwedge(m) \right),$$

where \mathcal{F}_U is the sheaf of holomorphic functions on an open set $U \subset \mathbb{C}^n$ and $\bigwedge(m)$ is the exterior (or Grassmann) algebra with m generators. A *complex analytic supermanifold* of dimension $n|m$ is a \mathbb{Z}_2 -graded ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ be a complex analytic supermanifold and

$$\mathcal{I}_{\mathcal{M}} = (\mathcal{O}_{\mathcal{M}})_{\bar{1}} + (\mathcal{O}_{\mathcal{M}})_{\bar{1}}^2$$

be the subsheaf of ideals generated by odd elements in $\mathcal{O}_{\mathcal{M}}$. We put $\mathcal{F}_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}}/\mathcal{I}_{\mathcal{M}}$. Then $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}})$ is a usual complex analytic manifold. It is called the *reduction* or *underlying space* of \mathcal{M} . We will write \mathcal{M}_0 instead of $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}})$ for simplicity of notation. Morphisms of supermanifolds are just morphisms of the corresponding \mathbb{Z}_2 -graded ringed spaces. If $f : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of supermanifolds, then we denote by f_0 the morphism of the underlying spaces $\mathcal{M}_0 \rightarrow \mathcal{N}_0$ and by f^* the morphism of the structure sheaves $\mathcal{O}_{\mathcal{N}} \rightarrow (f_0)_*(\mathcal{O}_{\mathcal{M}})$. If $x \in \mathcal{M}_0$ and \mathfrak{m}_x is the maximal ideal of the local superalgebra $(\mathcal{O}_{\mathcal{M}})_x$, then the vector superspace $T_x(\mathcal{M}) := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is the tangent space of \mathcal{M} at $x \in \mathcal{M}_0$.

Denote by $\mathcal{T}_{\mathcal{M}}$ the *tangent sheaf* or the *sheaf of vector fields* of \mathcal{M} . In other words, $\mathcal{T}_{\mathcal{M}}$ is the sheaf of derivations of the structure sheaf $\mathcal{O}_{\mathcal{M}}$. Since the sheaf $\mathcal{O}_{\mathcal{M}}$ is \mathbb{Z}_2 -graded, the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is also \mathbb{Z}_2 -graded, i.e. there is the natural decomposition $\mathcal{T}_{\mathcal{M}} = (\mathcal{T}_{\mathcal{M}})_{\bar{0}} \oplus (\mathcal{T}_{\mathcal{M}})_{\bar{1}}$, where

$$(\mathcal{T}_{\mathcal{M}})_{\bar{i}} := \left\{ v \in \mathcal{T}_{\mathcal{M}} \mid v((\mathcal{O}_{\mathcal{M}})_{\bar{j}}) \subset (\mathcal{O}_{\mathcal{M}})_{\bar{j}+\bar{i}} \right\}.$$

Let \mathcal{M}_0 be a complex analytic manifold and let \mathcal{E} be the sheaf of holomorphic sections of a vector bundle over \mathcal{M}_0 . Then the ringed space $(\mathcal{M}_0, \bigwedge \mathcal{E})$ is a supermanifold. In this case $\dim \mathcal{M} = n|m$, where $n = \dim \mathcal{M}_0$ and m is the rank of the locally free sheaf \mathcal{E} .

Definition 1. A supermanifold $(\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ is called *split* if $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ for a locally free sheaf \mathcal{E} on \mathcal{M}_0 . The grading of $\mathcal{O}_{\mathcal{M}}$ induces by an isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ and the natural \mathbb{Z} -grading of $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$ is called *split grading*.

For example, all smooth supermanifolds are split by Batchelor's Theorem. In [4] it was shown that all complex analytic Lie supergroups are split too. In this paper we study the splitting problem for complex analytic homogeneous supermanifolds.

3 Lie supergroups and their homogeneous spaces

3.1 Lie supergroups and super Harish-Chandra pairs.

A *Lie supergroup* is a group object in the category of supermanifolds, i.e. it is a supermanifold \mathcal{G} with three morphisms: the multiplication morphism, the inversion morphism and the identity morphism, which satisfy the usual conditions, modeling the group axioms. In this case the underlying space \mathcal{G}_0 is a Lie group. The structure sheaf of a (complex analytic) Lie supergroup can be explicitly described in terms of the corresponding Lie superalgebra and underlying Lie group using super Harish-Chandra pairs (see [5] and [14] for more details). Let us describe this construction briefly.

Definition 2. A *super Harish-Chandra pair* is a pair $(\mathcal{G}_0, \mathfrak{g})$ that consists of a Lie group \mathcal{G}_0 and a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0 = \text{Lie } \mathcal{G}_0$ provided with a representation $\text{Ad} : \mathcal{G}_0 \rightarrow \text{Aut } \mathfrak{g}$ of \mathcal{G}_0 in \mathfrak{g} such that:

- Ad preserves the parity and induces the adjoint representation of \mathcal{G}_0 on \mathfrak{g}_0 ;
- the differential $(d \text{Ad})_e$ at the identity $e \in \mathcal{G}_0$ coincides with the adjoint representation ad of \mathfrak{g}_0 on \mathfrak{g} .

If a super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g})$ is given, it determines the Lie supergroup \mathcal{G} in the following way, see [5]. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping superalgebra of \mathfrak{g} . It is clear that $\mathfrak{U}(\mathfrak{g})$ is a $\mathfrak{U}(\mathfrak{g}_0)$ -module, where $\mathfrak{U}(\mathfrak{g}_0)$ is the universal enveloping algebra of \mathfrak{g}_0 . Recall that we denote by $\mathcal{F}_{\mathcal{G}_0}$ the structure sheaf of the manifold \mathcal{G}_0 . The natural action of \mathfrak{g}_0 on the sheaf $\mathcal{F}_{\mathcal{G}_0}$ gives rise to a structure of $\mathfrak{U}(\mathfrak{g}_0)$ -module on $\mathcal{F}_{\mathcal{G}_0}(U)$ for any open set $U \subset \mathcal{G}_0$. Putting

$$\mathcal{O}_{\mathcal{G}}(U) = \text{Hom}_{\mathfrak{U}(\mathfrak{g}_0)}(\mathfrak{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}(U))$$

for every open $U \subset \mathcal{G}_0$, we get a sheaf $\mathcal{O}_{\mathcal{G}}$ of \mathbb{Z}_2 -graded vector spaces. (Here we assume that the functions in $\mathcal{F}_{\mathcal{G}_0}(U)$ are even.) The enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$ has a Hopf superalgebra structure. Using this structure we can define the product of elements from $\mathcal{O}_{\mathcal{G}}$ such that $\mathcal{O}_{\mathcal{G}}$ becomes a sheaf of superalgebras, see [5] and [14] for details. A supermanifold structure on $\mathcal{O}_{\mathcal{G}}$ is determined by the isomorphism $\Phi_{\mathfrak{g}} : \mathcal{O}_{\mathcal{G}} \rightarrow \text{Hom}(\bigwedge(\mathfrak{g}_1), \mathcal{F}_{\mathcal{G}_0}), f \mapsto f \circ \gamma_{\mathfrak{g}}$, where

$$\gamma_{\mathfrak{g}} : \bigwedge(\mathfrak{g}_1) \rightarrow \mathfrak{U}(\mathfrak{g}), \quad X_1 \wedge \cdots \wedge X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{|\sigma|} X_{\sigma(1)} \cdots X_{\sigma(r)}. \quad (1)$$

The following formulas define the multiplication morphism, the inversion morphism and the identity morphism respectively:

$$\begin{aligned}\mu^*(f)(X \otimes Y)(g, h) &= f(\text{Ad}(h^{-1})(X) \cdot Y)(gh); \\ \iota^*(f)(X)(g) &= f(\text{Ad}(g)(S(X)))(g^{-1}); \\ \varepsilon^*(f) &= f(1)(e).\end{aligned}\tag{2}$$

Here $X, Y \in \mathfrak{U}(\mathfrak{g})$, $f \in \mathcal{O}_{\mathcal{G}}$, $g, h \in \mathcal{G}_0$ and S is the antipode map of the Hopf superalgebra $\mathfrak{U}(\mathfrak{g})$. Here we identify the enveloping superalgebra $\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$ with the tensor product $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$.

Sometimes we will identify the Lie superalgebra \mathfrak{g} of a Lie supergroup \mathcal{G} with the tangent space $T_e(\mathcal{G})$ at $e \in \mathcal{G}_0$. The corresponding to $T \in T_e(\mathcal{G})$ left invariant vector field on \mathcal{G} is given by

$$(\text{id} \otimes T) \circ \mu^*, \tag{3}$$

where μ is the multiplication morphism of \mathcal{G} . (Recall that a vector field Y on \mathcal{G} is called *left invariant* if $(\text{id} \otimes Y) \circ \mu^* = \mu^* \circ Y$.) Denote by l_g and by r_g the left and right translations with respect to $g \in \mathcal{G}_0$, respectively. The morphisms l_g and r_g are given by the following formulas:

$$l_g^*(f)(X)(h) = f(X)(gh); \quad r_g^*(f)(X)(h) = f(\text{Ad}(g^{-1})X)(hg), \tag{4}$$

where $f \in \mathcal{O}_{\mathcal{G}}$, $X \in \mathfrak{U}(\mathfrak{g})$ and $g, h \in \mathcal{G}_0$.

3.2 Homogeneous supermanifolds.

An *action of a Lie supergroup \mathcal{G} on a supermanifold \mathcal{M}* is a morphism $\nu : \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ such that the usual conditions modeling group action axioms hold. Any vector $X \in T_e(\mathcal{G})$ defines the vector field on \mathcal{M} by the following formula:

$$X \mapsto (X \otimes \text{id}) \circ \nu^*. \tag{5}$$

Definition 3. An action ν is called *transitive* if ν_0 is a transitive action of the Lie group \mathcal{G}_0 on \mathcal{M}_0 and the vector fields (5) generates the tangent space $T_x(\mathcal{M})$ at any point $x \in \mathcal{M}_0$. In this case the supermanifold \mathcal{M} is called *\mathcal{G} -homogeneous*. A supermanifold \mathcal{M} is called *homogeneous*, if it possesses a transitive action of a Lie supergroup.

If a supermanifold \mathcal{M} is \mathcal{G} -homogeneous and $\nu : \mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ is the corresponding transitive action, then \mathcal{M} is isomorphic to the supermanifold \mathcal{G}/\mathcal{H} , where \mathcal{H} is the isotropy subsupergroup of a certain point (see [16] for

details). Recall that the underlying space of \mathcal{G}/\mathcal{H} is the complex analytic manifold $\mathcal{G}_0/\mathcal{H}_0$ and the structure sheaf $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$ of \mathcal{G}/\mathcal{H} is given by

$$\mathcal{O}_{\mathcal{G}/\mathcal{H}} = \left\{ f \in (\pi_0)_*(\mathcal{O}_{\mathcal{G}}) \mid \mu_{\mathcal{G} \times \mathcal{H}}^*(f) = \text{pr}^*(f) \right\}, \quad (6)$$

where $\pi_0 : \mathcal{G}_0 \rightarrow \mathcal{G}_0/\mathcal{H}_0$ is the natural map, $\mu_{\mathcal{G} \times \mathcal{H}}$ is the restriction of the multiplication map on $\mathcal{G} \times \mathcal{H}$ and $\text{pr} : \mathcal{G} \times \mathcal{H} \rightarrow \mathcal{G}$ is the natural projection. Using (2) we can rewrite the condition $\mu_{\mathcal{G} \times \mathcal{H}}^*(f) = \text{pr}^*(f)$ in the following way:

$$f\left(\text{Ad}(h^{-1})(X)Y\right)(gh) = \begin{cases} f(X)(g), & Y \in \mathbb{C}; \\ 0, & Y \notin \mathbb{C}; \end{cases} \quad (7)$$

where $X \in \mathfrak{u}(\mathfrak{g})$, $Y \in \mathfrak{u}(\mathfrak{h})$, $\mathfrak{h} = \text{Lie } \mathcal{H}$, $g \in \mathcal{G}_0$ and $h \in \mathcal{H}_0$.

Let $Y \in \mathfrak{g}$ and $f \in \mathcal{O}_{\mathcal{G}}$. Then the operator defined by the formula

$$Y(f)(X) = (-1)^{p(Y)} f(XY), \quad (8)$$

where $p(Y)$ is the parity of Y , is a left invariant vector field on \mathcal{G} . From (4), (7) and (8) it follows that

$f \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}$ if and only if f is \mathcal{H}_0 -right invariant, i.e. $r_h^*(f) = f$ for any $h \in \mathcal{H}_0$, and $Y(f) = 0$ for all $Y \in \mathfrak{h}_{\bar{1}}$, where $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$.

Sometimes we will consider also the left action $\mathcal{H} \times \mathcal{G} \rightarrow \mathcal{G}$ of a subsupergroup \mathcal{H} on a Lie supergroup \mathcal{G} . The corresponding quotient supermanifold we will denote by $\mathcal{H} \backslash \mathcal{G}$.

3.3 More about split supermanifolds.

Recall that a supermanifold \mathcal{M} is called split if its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$, where \mathcal{E} is a locally free sheaf on \mathcal{M}_0 . In this case, $\mathcal{O}_{\mathcal{M}}$ possesses the \mathbb{Z} -grading induced by the natural \mathbb{Z} -grading of $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$ and by isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$. Such gradings of $\mathcal{O}_{\mathcal{M}}$ we call split.

Proposition 1. *Any Lie supergroup \mathcal{G} is split.*

This statement follows from the fact that any Lie supergroup is determined by its super Harish-Chandra pair. A different proof of this result (probably the first one) was given in [4]. For completeness we give here another proof.

Proof. The underlying space \mathcal{G}_0 is a closed Lie subsupergroup of \mathcal{G} . Hence, there exists the homogeneous space $\mathcal{G}/\mathcal{G}_0$, which is isomorphic to the supermanifold \mathcal{N} such that \mathcal{N}_0 is a point $\text{pt} = \mathcal{G}_0/\mathcal{G}_0$ and $\mathcal{O}_{\mathcal{N}} \simeq \bigwedge(m)$, where $m = \dim \mathfrak{g}_{\bar{1}}$. By definition, the structure sheaf $\mathcal{O}_{\mathcal{N}}$ consists of all

r_g -invariant functions, $g \in \mathcal{G}_0$. We have the natural map $\varphi : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{G}_0$, where $\varphi_0 : \mathcal{G}_0 \rightarrow \text{pt}$ and $\varphi^* : \mathcal{O}_{\mathcal{N}} \rightarrow (\varphi_0)_*(\mathcal{O}_{\mathcal{G}})$ is the inclusion. It is known that $\varphi : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{G}_0$ is a principal bundle (see [16]). Using the fact that the underlying space of $\mathcal{G}/\mathcal{G}_0$ is a point we get $\mathcal{G} \simeq \mathcal{N} \times \mathcal{G}_0$. Note that this is an isomorphism of supermanifolds but not of Lie supergroups. \square

Example 1. As an example of a homogeneous non-split supermanifold we can cite the super-grassmannian $\mathbf{Gr}_{m|n,r|s}$ for $0 < r < m$ and $0 < s < n$. Super-grassmannians of other types are split (see Example 3).

Denote by \mathbf{SSM} the category of split supermanifolds. Objects $\text{Ob } \mathbf{SSM}$ in this category are all split supermanifolds \mathcal{M} with fixed split gradings. Further if $X, Y \in \text{Ob } \mathbf{SSM}$, we put

$$\text{Hom}(X, Y) = \left\{ \begin{array}{l} \text{morphisms from } X \text{ to } Y \\ \text{preserving the split gradings} \end{array} \right\}$$

As in the category of supermanifolds, we can define in \mathbf{SSM} a group object (*split Lie supergroup*), an action of a split Lie supergroup on a split supermanifold (*split action*) and a *split homogeneous supermanifold*.

There is a functor gr from the category of supermanifolds to the category of split supermanifolds. Let us briefly describe this construction. Let \mathcal{M} be a supermanifold. Denote by $\mathcal{J}_{\mathcal{M}} \subset \mathcal{O}_{\mathcal{M}}$ the subsheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. Then by definition $\text{gr } \mathcal{M} = (\mathcal{M}_0, \text{gr } \mathcal{O}_{\mathcal{M}})$ is the split supermanifold with the structure sheaf

$$\text{gr } \mathcal{O}_{\mathcal{M}} = \bigoplus_{p \geq 0} (\text{gr } \mathcal{O}_{\mathcal{M}})_p, \quad \mathcal{J}_{\mathcal{M}}^0 := \mathcal{O}_{\mathcal{M}}, \quad (\text{gr } \mathcal{O}_{\mathcal{M}})_p := \mathcal{J}_{\mathcal{M}}^p / \mathcal{J}_{\mathcal{M}}^{p+1}.$$

In this case $(\text{gr } \mathcal{O}_{\mathcal{M}})_1$ is a locally free sheaf and there is a natural isomorphism of $\text{gr } \mathcal{O}_{\mathcal{M}}$ onto $\bigwedge(\text{gr } \mathcal{O}_{\mathcal{M}})_1$. If $\psi = (\psi_0, \psi^*) : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism, then $\text{gr}(\psi) = (\psi_0, \text{gr}(\psi^*)) : \text{gr } \mathcal{M} \rightarrow \text{gr } \mathcal{N}$ is defined by

$$\text{gr}(\psi^*)(f + \mathcal{J}_{\mathcal{N}}^p) := \psi^*(f) + \mathcal{J}_{\mathcal{M}}^p \text{ for } f \in (\mathcal{J}_{\mathcal{N}})^{p-1}.$$

Recall that by definition every morphism of supermanifolds is even and as a consequence sends $\mathcal{J}_{\mathcal{N}}^p$ into $\mathcal{J}_{\mathcal{M}}^p$.

3.4 Split Lie supergroups.

Let \mathcal{G} be a Lie supergroup with the supergroup morphisms μ , ι and ε : the multiplication, the inversion and the identity morphism, respectively. In this section we assign three split Lie supergroups \mathcal{G}^1 , \mathcal{G}^2 and \mathcal{G}^3 to \mathcal{G} and we show that these split Lie supergroups are pairwise isomorphic.

(1) The construction of \mathcal{G}^1 is very simple: we just apply functor gr to \mathcal{G} . Clearly, $\mathcal{G}^1 := \text{gr } \mathcal{G}$ is a split Lie supergroup with the supergroup morphisms $\text{gr}(\mu)$, $\text{gr}(\iota)$ and $\text{gr}(\varepsilon)$.

(2) Consider the super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g}^2)$, where \mathfrak{g}^2 is the following Lie superalgebra: \mathfrak{g}^2 and \mathfrak{g} are isomorphic as vector superspaces and the Lie bracket in \mathfrak{g}^2 is defined by the following formula:

$$[X, Y]_{\mathfrak{g}^2} = \begin{cases} [X, Y]_{\mathfrak{g}}, & \text{if } X, Y \in \mathfrak{g}_0 \text{ or } X \in \mathfrak{g}_0 \text{ and } Y \in \mathfrak{g}_1; \\ 0, & \text{if } X, Y \in \mathfrak{g}_1. \end{cases} \quad (9)$$

Denote by \mathcal{G}^2 the Lie supergroup corresponding to $(\mathcal{G}_0, \mathfrak{g}^2)$.

(3) Consider the sheaf $\mathcal{O}_{\mathcal{G}^3} := \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0})$. For the ringed space $\mathcal{G}^3 := (\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})$ we can repeat the construction from Section 3.1. Indeed, this ringed space is clearly a supermanifold. Furthermore, the exterior algebra $\bigwedge \mathfrak{g}_1$ is also a Hopf algebra. Therefore, we can define on \mathcal{G}^3 the multiplication, the inversion and the identity morphisms respectively by the following formulas:

$$\begin{aligned} (\mu^3)^*(f)(X \wedge Y)(g, h) &= f(\text{Ad}(h^{-1})(X) \wedge Y)(gh); \\ (\iota^3)^*(f)(X)(g) &= f(\text{Ad}(g)(S'(X)))(g^{-1}); \\ (\varepsilon^3)^*(f) &= f(1)(e). \end{aligned} \quad (10)$$

Here $X, Y \in \bigwedge \mathfrak{g}_1$, $f \in \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0})$, $g, h \in \mathcal{G}_0$ and S' is the antipode map of the Hopf superalgebra $\bigwedge \mathfrak{g}_1$. Hence, $\mathcal{G}^3 := (\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})$ is a Lie supergroup. Since

$$\text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0}) = \bigoplus_{p \geq 0} \text{Hom}_{\mathbb{C}}(\bigwedge^p \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0})$$

is \mathbb{Z} -graded and the morphisms (10) preserve this \mathbb{Z} -grading, we see that \mathcal{G}^3 is a split Lie supergroup.

Later on we will need the explicit expression of left and right translations l'_g and r'_g in \mathcal{G}^3 :

$$(l'_g)^*(f)(X)(h) = f(X)(gh); \quad (r'_g)^*(f)(X)(h) = f(\text{Ad}(g^{-1})X)(hg), \quad (11)$$

where $f \in \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0})$, $X \in \bigwedge \mathfrak{g}_1$ and $g, h \in \mathcal{G}_0$.

In fact, all these split Lie supergroups are isomorphic. To show this we need the following lemma:

Lemma 1. *Let \mathfrak{k} be a Lie superalgebra and $X_i, Y_j \in \mathfrak{k}_1$, $i = 1, \dots, r$, $j = 1, \dots, s$ be any elements. Assume that $[X_i, Y_j] = 0$ for any i, j . Then we have*

$$\gamma_{\mathfrak{k}}(X_1 \wedge \dots \wedge X_r \wedge Y_1 \wedge \dots \wedge Y_s) = \gamma_{\mathfrak{k}}(X_1 \wedge \dots \wedge X_r) \cdot \gamma_{\mathfrak{k}}(Y_1 \wedge \dots \wedge Y_s),$$

where $\gamma_{\mathfrak{k}}$ is given by (1).

Proof. A direct calculation. \square

Proposition 2. *We have $\mathcal{G}^1 \simeq \mathcal{G}^2 \simeq \mathcal{G}^3$ in the category of Lie supergroups.*

Proof. (a) The statement $\mathcal{G}^1 \simeq \mathcal{G}^2$ was proven in [15], Theorem 3.

(b) Let us show that $\mathcal{G}^2 \simeq \mathcal{G}^3$. Applying Lemma 1 to \mathfrak{g}^2 and to any elements $X_i, Y_j \in \mathfrak{g}_1^2$, we see that in this case $\gamma_{\mathfrak{g}^2}$ is not only isomorphism of super coalgebras but of Hopf superalgebras. In other words, the isomorphism

$$\Phi_{\mathfrak{g}^2} : \text{Hom}_{\mathfrak{U}(\mathfrak{g}_0^2)}(\mathfrak{U}(\mathfrak{g}^2), \mathcal{F}_{\mathcal{G}_0}) \rightarrow \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0})$$

is an isomorphism of Lie supergroups. \square

4 Split grading operators

Let again \mathcal{M} be a supermanifold, $\text{gr } \mathcal{M}$ be the corresponding split supermanifold and \mathcal{J} be the sheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. We denote by $\mathcal{T} = \text{Der } \mathcal{O}_{\mathcal{M}}$ and by $\text{gr } \mathcal{T} = \text{Der}(\mathcal{O}_{\text{gr } \mathcal{M}})$ the tangent sheaf of \mathcal{M} and of $\text{gr } \mathcal{M}$, respectively. The sheaf \mathcal{T} is naturally \mathbb{Z}_2 -graded and the sheaf $\text{gr } \mathcal{T}$ is naturally \mathbb{Z} -graded: the gradings are induced by the \mathbb{Z}_2 and \mathbb{Z} -grading of $\mathcal{O}_{\mathcal{M}}$ and $\text{gr } \mathcal{O}_{\mathcal{M}}$, respectively. In other words, we have the decomposition:

$$\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1, \quad \text{gr } \mathcal{T} = \bigoplus_{p \geq -1} (\text{gr } \mathcal{T})_p.$$

The sheaves \mathcal{T} and $\text{gr } \mathcal{T}$ are related: this relation can be expressed by the following exact sequence:

$$0 \longrightarrow \mathcal{T}_{(2)\bar{0}} \longrightarrow \mathcal{T}_0 \xrightarrow{\alpha} (\text{gr } \mathcal{T})_0 \longrightarrow 0, \quad (12)$$

where

$$\mathcal{T}_{(2)\bar{0}} = \{v \in \mathcal{T}_0 \mid v(\mathcal{O}_{\mathcal{M}}) \subset \mathcal{J}^2\}.$$

The morphism α in (12) is the composition of the natural morphism $\mathcal{T}_0 \rightarrow \mathcal{T}_0/\mathcal{T}_{(2)\bar{0}}$ and the isomorphism $\mathcal{T}_0/\mathcal{T}_{(2)\bar{0}} \rightarrow (\text{gr } \mathcal{T})_0$ that is given by

$$[w] \longmapsto \tilde{w}, \quad \tilde{w}(f + \mathcal{J}^{p+1}) := w(f) + \mathcal{J}^{p+1},$$

where $w \in \mathcal{T}_0$, $[w]$ is the image of w in $\mathcal{T}_0/\mathcal{T}_{(2)\bar{0}}$ and $f \in \mathcal{J}^p$.

Assume that the sheaf $\mathcal{O}_{\mathcal{M}}$ is \mathbb{Z} -graded, i.e. $\mathcal{O}_{\mathcal{M}} = \bigoplus_p (\mathcal{O}_{\mathcal{M}})_p$. Then we have the map $w : \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}}$ defined by $w(f) = pf$, where $f \in (\mathcal{O}_{\mathcal{M}})_p$. Such maps are called *grading operators* on \mathcal{M} .

Definition 4. We call a grading operator w on \mathcal{M} a *split grading operator* if it corresponds to a split grading of $\mathcal{O}_{\mathcal{M}}$, see Definition 1.

In fact any split grading operator w on \mathcal{M} is an even vector field on \mathcal{M} . Indeed, w is linear, it preserves the parity in $\mathcal{O}_{\mathcal{M}}$ and for $f \in (\mathcal{O}_{\mathcal{M}})_p$ and $g \in (\mathcal{O}_{\mathcal{M}})_q$ we have:

$$w(fg) = (p+q)fg = (pf)g + f(qg) = w(f)g + fw(g).$$

Note that $fg \in (\mathcal{O}_{\mathcal{M}})_{p+q}$.

By definition the sheaf $\text{gr } \mathcal{O}_{\mathcal{M}}$ is \mathbb{Z} -graded. Denote by a the corresponding split grading operator.

Lemma 2. 1. A supermanifold \mathcal{M} is split if and only if the vector field a is contained in $\text{Im } H^0(\alpha)$, where

$$H^0(\alpha) : H^0(\mathcal{M}_0, \mathcal{T}_0) \rightarrow H^0(\mathcal{M}_0, (\text{gr } \mathcal{T})_0).$$

(We applied the functor $H^0(\mathcal{M}_0, -)$ to the sequence (12). We write $H^0(\alpha)$ instead of $H^0(\mathcal{M}_0, \alpha)$ for notational simplicity.)

2. If w is a split grading operator on \mathcal{M} , then any other split grading operator on \mathcal{M} has the form $w + \chi$, where $\chi \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}})$.

Proof. 1. The statement of the lemma can be deduced from the following observation made by Koszul in [4, Lemma 1.1 and Section 3]. Let A be a commutative superalgebra over \mathbb{C} and \mathfrak{m} be a nilpotent ideal in A . An even derivation w of A is called *adapted to the filtration*

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \dots$$

if

$$(w - r \text{id})(\mathfrak{m}^r) \subset \mathfrak{m}^{r+1} \text{ for any } r \geq 0.$$

Denote by $D_{\mathfrak{m}}^{ad}$ the set of all derivations adapted to \mathfrak{m} . In [4, Lemma 1.1] it was shown that $D_{\mathfrak{m}}^{ad}$ is not empty if and only if the filtration of A is splittable. Moreover, if $w \in D_{\mathfrak{m}}^{ad}$, then the corresponding splitting of A is given by eigenspaces of the derivation w : $A = \bigoplus_i A_i$, where A_i is the eigenspace of w with the eigenvalue i , and $\mathfrak{m}^r = A_r \oplus \mathfrak{m}^{r+1}$ for all $r \geq 0$.

We apply Koszul's observation to the sheaf of superalgebras $\mathcal{O}_{\mathcal{M}}$ and its subsheaf of ideals \mathcal{J} . The set $D_{\mathcal{J}}^{ad}$ is in this case the set of global derivations of $\mathcal{O}_{\mathcal{M}}$ adapted to the filtration

$$\mathcal{O}_{\mathcal{M}} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \dots \tag{13}$$

Clearly, $D_{\mathcal{J}}^{ad}$ is not empty if and only if a is contained in $\text{Im } H^0(\alpha)$. (Actually, $H^0(\alpha)(D_{\mathcal{J}}^{ad}) = a$.) Furthermore, if the supermanifold \mathcal{M} is split, i.e.

we have a split grading $\mathcal{O}_{\mathcal{M}} = \bigoplus_{p \geq 0} (\mathcal{O}_{\mathcal{M}})_p$, then $\mathcal{J}^q = \bigoplus_{p \geq q} (\mathcal{O}_{\mathcal{M}})_p$ and $\mathcal{J}^q = (\mathcal{O}_{\mathcal{M}})_q \oplus \mathcal{J}^{q+1}$. Hence, the split grading determine the splitting of the filtration (13) and the corresponding split grading operator belongs to $D_{\mathcal{J}}^{ad}$.

Conversely, if there exists $w \in D_{\mathcal{J}}^{ad}$, then we can decompose the sheaf $\mathcal{O}_{\mathcal{M}}$ into eigenspaces

$$(\mathcal{O}_{\mathcal{M}})_q := \{f \in \mathcal{O}_{\mathcal{M}} | w(f) = qf\}.$$

In this case the sheaves $\bigoplus_p (\mathcal{O}_{\mathcal{M}})_p$ and $\text{gr } \mathcal{O}_{\mathcal{M}}$ are isomorphic as \mathbb{Z} -graded sheaves of superalgebras since $\mathcal{J}^q = (\mathcal{O}_{\mathcal{M}})_q \oplus \mathcal{J}^{q+1}$. Hence, the supermanifold is split.

2. Applying the left-exact functor $H^0(\mathcal{M}_0, -)$ to (12), we get the following exact sequence:

$$0 \longrightarrow H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}}) \longrightarrow H^0(\mathcal{M}_0, \mathcal{T}_{\bar{0}}) \xrightarrow{H^0(\alpha)} H^0(\mathcal{M}_0, (\text{gr } \mathcal{T})_0).$$

If w_1, w_2 are two split grading operators on \mathcal{M} , then

$$H^0(\alpha)(w_1) = H^0(\alpha)(w_2) = a,$$

according to the part 1. Therefore, $w_1 - w_2 \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}})$. The result follows. \square

Example 2. Consider the supermanifold $\mathcal{G}_0 \setminus \mathcal{G}$. Its structure sheaf is isomorphic to $\bigwedge(\mathfrak{g}_{\bar{1}})$ (compare with Example 1). Denote by (ε_i) the system of odd (global) coordinates on $\mathcal{G}_0 \setminus \mathcal{G}$. An example of a split grading operator on the Lie supergroup \mathcal{G} is $\sum \varepsilon^i X_i$. Here (X_i) is a basis of odd left invariant vector fields on \mathcal{G} such that $X_i(\varepsilon^j)(e) = \delta_i^j$. We may produce other examples if we use right invariant vector fields or odd (global) coordinates on $\mathcal{G}/\mathcal{G}_0$.

By Lemma 2, any split grading operator on a Lie supergroup \mathcal{G} is given by $\sum \varepsilon^i X_i + \chi$, where $\chi \in H^0(\mathcal{G}_0, \mathcal{T}_{(2)\bar{0}})$ is any vector field on \mathcal{G} .

5 Compatible split gradings on \mathcal{G}/\mathcal{H}

5.1 Compatible gradings on \mathcal{G}/\mathcal{H} .

Let \mathcal{G} be a Lie supergroup and $\mathcal{M} = \mathcal{G}/\mathcal{H}$ be a homogeneous supermanifold. As above we denote by $\pi : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ the natural projection.

Definition 5. A split grading of the sheaf $\mathcal{O}_{\mathcal{G}} = \bigoplus_p (\mathcal{O}_{\mathcal{G}})_p$ is called *compatible* with the inclusion $\mathcal{O}_{\mathcal{M}} \subset (\pi_0)_*(\mathcal{O}_{\mathcal{G}})$ if the following holds:

$$f \in \mathcal{O}_{\mathcal{M}} \Rightarrow f_p \in \mathcal{O}_{\mathcal{M}} \text{ for all } p,$$

where $f = \sum f_p$ and $f_p \in (\pi_0)_*((\mathcal{O}_{\mathcal{G}})_p)$.

Let us take any split grading operator w on \mathcal{G} . Clearly, the corresponding split grading of $\mathcal{O}_{\mathcal{G}}$ is compatible with $\mathcal{O}_{\mathcal{M}}$ if and only if $w(\mathcal{O}_{\mathcal{M}}) \subset \mathcal{O}_{\mathcal{M}}$. It is not clear from Definition 5 that the compatible grading

$$(\mathcal{O}_{\mathcal{M}})_p = \mathcal{O}_{\mathcal{M}} \cap (\pi_0)_*((\mathcal{O}_{\mathcal{G}})_p) \quad (14)$$

of $\mathcal{O}_{\mathcal{M}}$, if it exists, is a split grading of $\mathcal{O}_{\mathcal{M}}$. However, the following proposition holds:

Proposition 3. *Assume that we have the \mathbb{Z} -grading:*

$$\mathcal{O}_{\mathcal{M}} = \bigoplus_{p \geq 0} (\mathcal{O}_{\mathcal{M}})_p,$$

where $(\mathcal{O}_{\mathcal{M}})_p$ are as in (14). Then this grading is a split grading.

Proof. The idea of the proof is to apply Lemma 2 to the grading operator $w' := w|_{\mathcal{O}_{\mathcal{M}}}$ on \mathcal{M} . Denote by $\mathcal{J}_{\mathcal{M}}$ and by $\mathcal{J}_{\mathcal{G}}$ the sheaves of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{O}_{\mathcal{G}}$, respectively. Our aim is to show that

$$w'(f) + \mathcal{J}_{\mathcal{M}}^{p+1} = pf + \mathcal{J}_{\mathcal{M}}^{p+1},$$

where $f \in \mathcal{J}_{\mathcal{M}}^p$. In other words, we want to show that $H^0(\alpha)(w')$ is a split grading operator for the grading of $\text{gr } \mathcal{O}_{\mathcal{M}}$. (We use notations of Lemma 2.) We have:

$$\begin{aligned} (\text{gr } \pi)^*(w'(f) + \mathcal{J}_{\mathcal{M}}^{p+1}) &= w(f) + \mathcal{J}_{\mathcal{G}}^{p+1} = pf + \mathcal{J}_{\mathcal{G}}^{p+1}; \\ (\text{gr } \pi)^*(pf + \mathcal{J}_{\mathcal{M}}^{p+1}) &= pf + \mathcal{J}_{\mathcal{G}}^{p+1}. \end{aligned}$$

Since the map $(\text{gr } \pi)^*$ is injective, we get, $w'(f) + \mathcal{J}_{\mathcal{M}}^{p+1} = pf + \mathcal{J}_{\mathcal{M}}^{p+1}$. \square

5.2 \mathcal{H} -invariant split grading operators.

First of all let us consider the situation when a split grading operator w on \mathcal{G} is invariant with respect to a Lie supersubgroup \mathcal{H} . In terms of super Harish-Chandra pairs this means:

$$\begin{aligned} r_h^* \circ w &= w \circ r_h^*, & \text{for all } h \in \mathcal{H}_0; \\ [Y, w] &= 0, & \text{for all } Y \in \mathfrak{h}_{\bar{1}}. \end{aligned} \quad (15)$$

Here $(\mathcal{H}_0, \mathfrak{h})$ is the super Harish-Chandra pair of \mathcal{H} , r_h is the right translation and Y is an odd left invariant vector field.

Proposition 4. *Assume that w is an \mathcal{H} -invariant split grading operator on \mathcal{G} , i.e. equations (15) hold. Then \mathcal{H} is an ordinary Lie group.*

Proof. The idea of the proof is to show that the Lie superalgebra \mathfrak{h} of \mathcal{H} has the trivial odd part: $\mathfrak{h}_{\bar{1}} = \{0\}$.

In Example 2 we saw that any split grading operator on \mathcal{G} is given by $w = \sum \varepsilon^i X_i + \chi$. If Z is a vector field on \mathcal{G} , denote by $Z_e \in T_e(\mathcal{G})$ the corresponding tangent vector at the identity $e \in \mathcal{G}_0$. Consider the second equation in (15). At the point e , we have

$$[Y, w]_e = \left(\sum_i Y(\varepsilon^i) X_i - \sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y + [Y, \chi] \right)_e = 0$$

for any $Y \in \mathfrak{h}_{\bar{1}}$. Furthermore,

$$\left(\sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y \right)_e = 0 \quad \text{and} \quad [Y, \chi]_e = 0,$$

because $\varepsilon^i(e) = 0$ and because $\chi \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}})$. Therefore,

$$[Y, w]_e = \sum_i Y(\varepsilon^i)(e)(X_i)_e = 0$$

The tangent vectors $(X_i)_e$ form a basis in $T_e(\mathcal{G})_{\bar{1}}$, hence $Y(\varepsilon^i)(e) = 0$ for all i . The last statement is equivalent to $Y_e = 0$. Since Y is a left invariant vector field, we get $Y = 0$. The proof is complete. \square

Remark. It is well known that the supermanifold \mathcal{G}/\mathcal{H} , where \mathcal{H} is an ordinary Lie group, is split (see [5] or [14]). Therefore, the case of \mathcal{H} -invariant split grading operators does not lead to new examples of homogeneous split supermanifolds.

5.3 \mathcal{G}_0 -left invariant split grading operators.

Consider now a more general situation, when a split grading operator w leaves $\mathcal{O}_{\mathcal{M}}$ invariant. Let $f \in \mathcal{O}_{\mathcal{M}}$. Then $w(f) \in \mathcal{O}_{\mathcal{M}}$ if and only if

$$r_h^*(w(f)) = w(f) \quad \text{and} \quad Y(w(f)) = 0$$

for $h \in \mathcal{H}_0$ and $Y \in \mathfrak{h}_{\bar{1}}$. These conditions are equivalent to the following ones:

$$(r_h^* \circ w \circ (r_h^{-1})^* - w)|_{\mathcal{O}_{\mathcal{M}}} = 0; \quad [Y, w]|_{\mathcal{O}_{\mathcal{M}}} = 0. \quad (16)$$

Recall that $r_h^{-1} = r_{h^{-1}}$.

It seems to us that the system (16) is hard to solve in general. Consider now a special type of split grading operators, called \mathcal{G}_0 -left invariant grading operators.

Definition 6. A split grading of $\mathcal{O}_{\mathcal{G}}$ is called \mathcal{G}_0 -left invariant if it is invariant with respect to left translations. In other words, from $f \in (\mathcal{O}_{\mathcal{G}})_p$ it follows that $l_g^*(f) \in (\mathcal{O}_{\mathcal{G}})_p$ for all $g \in \mathcal{G}_0$.

It is easy to see that a split grading of $\mathcal{O}_{\mathcal{G}}$ is \mathcal{G}_0 -left invariant if and only if the corresponding split grading operator w is invariant with respect to left translations: $l_g^* \circ w = w \circ l_g^*$, $g \in \mathcal{G}_0$. For example, the split grading operator $\sum \varepsilon^i X_i$ constructed in Example 2 is a \mathcal{G}_0 -left invariant split grading operator, because ε^i are \mathcal{G}_0 -left invariant functions and X_i are left invariant vector fields. In this section we will describe all such operators.

In Section 3.4 we have seen that the supermanifold $(\mathcal{G}_0, \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}))$ is a Lie supergroup isomorphic to $\text{gr } \mathcal{G}$. We need the following lemma:

Lemma 3. *The map*

$$\begin{aligned} \Phi_{\mathfrak{g}} : \mathcal{O}_{\mathcal{G}} &\rightarrow \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}), \\ f &\mapsto f \circ \gamma_{\mathfrak{g}} \end{aligned}$$

from Section 3.1 is invariant with respect to left and right translations.

Proof. For any $h \in \mathcal{G}_0$, denote by r'_h and l'_h the right and the left translation in the Lie supergroup $\mathcal{G}^3 = (\mathcal{G}_0, \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}))$, respectively. (See, (11)) Let us show that

$$(r'_h)^* \circ \Phi_{\mathfrak{g}} = \Phi_{\mathfrak{g}} \circ r_h^*. \quad (17)$$

Let us take $Z \in \bigwedge \mathfrak{g}_{\bar{1}}$ and $g, h \in \mathcal{G}_0$. Using (4) we have

$$\begin{aligned} [(r'_h)^* \circ \Phi_{\mathfrak{g}}](f)(Z)(g) &= \Phi_{\mathfrak{g}}(f)(\text{Ad}(h^{-1})(Z))(gh) = \\ f(\gamma_{\mathfrak{g}}(\text{Ad}(h^{-1})(Z)))(gh) &= f(\text{Ad}(h^{-1})(\gamma_{\mathfrak{g}}(Z)))(gh) = \\ r_h^*(f)(\gamma_{\mathfrak{g}}(Z))(g) &= [\Phi_{\mathfrak{g}} \circ r_h^*](f)(Z)(g). \end{aligned}$$

Similarly, we get

$$(l'_h)^* \circ \Phi_{\mathfrak{g}} = \Phi_{\mathfrak{g}} \circ l_h^*.$$

□

The following observation is known to experts, but we cannot find it in the literature:

Lemma 4. *The space of \mathcal{G}_0 -left invariant vector fields $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ on a Lie supergroup \mathcal{G} is isomorphic to $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}$. The isomorphism is given by:*

$$f \otimes Z \xrightarrow{F} fZ,$$

where $f \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}})$ and $Z \in \mathfrak{g}$.

Proof. Clearly, the map F is injective and its image is contained in the vector space $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$. Let us show that any vector field v in $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ is contained in $\text{Im}(F)$.

Let (X_i) and (Z_j) be a basis of odd and even left invariant (with respect to the supergroup \mathcal{G}) vector fields on \mathcal{G} , respectively. Assume that

$$v = \sum f^i X_i + \sum g^j Z_j,$$

where $f^i, g^j \in H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}})$, be the decomposition of v with respect to this basis. We have:

$$\begin{aligned} l_g^* \circ v &= \sum l_g^*(f^i) l_g^* \circ X_i + \sum l_g^*(g^j) l_g^* \circ Z_j = \\ &= \sum l_g^*(f^i) X_i \circ l_g^* + \sum l_g^*(g^j) Z_j \circ l_g^* = v \circ l_g^*. \end{aligned}$$

Therefore, $l_g^*(f^i) = f^i$ and $l_g^*(g^j) = g^j$ for all $g \in \mathcal{G}_0$. In other words, $f^i, g^j \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}})$. The proof is complete. \square

The Lie supergroup \mathcal{G} acts on the vector superspace $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$. This action we can describe in terms of the corresponding super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g})$ in the following way:

$$g \mapsto (X \mapsto r_g^* \circ X \circ (r_g^{-1})^*), \quad Y \mapsto (X \mapsto [Y, X]), \quad (18)$$

where $g \in \mathcal{G}_0$, $X \in H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ and $Y \in \mathfrak{g}$. Note that this action is well-defined because \mathcal{G} -left and right actions on $H^0(\mathcal{G}_0, \mathcal{T})$ commute. The Lie supergroup \mathcal{G} acts also on the vector superspace $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}$. This action is given by right translations r_g^* on $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}})$ and by the formulas (18) on \mathfrak{g} if we assume that $X \in \mathfrak{g}$. Clearly, the isomorphism F from Lemma 4 is equivariant. From now on we will identify $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ and $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}$ via isomorphism F from Lemma 4.

If \mathcal{H} is a Lie subgroup of \mathcal{G} and $\mathfrak{h} = \text{Lie } \mathcal{H}$ then $\mathfrak{g}/\mathfrak{h}$ is an \mathcal{H} -module.

Lemma 5. *Let us take a \mathcal{G}_0 -left invariant split grading operator w . The vector field w satisfies (16) if and only if*

$$\bar{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}}, \quad (19)$$

where \bar{w} is the image of w by the natural mapping

$$H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g} \rightarrow H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h}.$$

Proof. Let $\bar{w} \in (H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}}$. It follows that

$$r_h^* \circ w \circ (r_h^{-1})^* - w \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{h}, \quad h \in \mathcal{H}_0,$$

and

$$[Y, w] \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{h}, \quad Y \in \mathfrak{h}.$$

Hence, the conditions (16) are satisfied.

On the other hand, if the conditions (16) are satisfied, then the vector fields $r_h^* \circ w \circ (r_h^{-1})^* - w$ and $[Y, w]$ are vertical with respect to the projection $\pi : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$. Therefore, $r_h^* \circ w \circ (r_h^{-1})^* - w$ and $[Y, w]$ belong to the superspace $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{h}$. It is equivalent to conditions (19). \square

Now our aim is to describe the space $(H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}_0}$. We have seen in Proposition 1 that the superspace $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}})$ is isomorphic to $\bigwedge \mathfrak{g}_1^*$. Actually this isomorphism can be chosen in \mathcal{G}_0 -equivariant way. More precisely, we need the following lemma.

Proposition 5. a. *We have*

$$\begin{aligned} H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g} &\simeq \bigwedge (\mathfrak{g}_1^*) \otimes \mathfrak{g} && \text{as } \mathcal{G}_0\text{-modules,} \\ H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h} &\simeq \bigwedge (\mathfrak{g}_1^*) \otimes \mathfrak{g}/\mathfrak{h} && \text{as } \mathcal{H}_0\text{-modules,} \end{aligned}$$

where the action of \mathcal{G}_0 on $\bigwedge (\mathfrak{g}_1^*)$ is standard.

b. *There exists a \mathcal{G}_0 -left and right invariant split grading operator on \mathcal{G} .*

Proof. a. We have to show that there exists an \mathcal{G}_0 -equivariant isomorphism

$$H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \xrightarrow{\beta} \bigwedge \mathfrak{g}_1^*.$$

Then the map $\beta \otimes \text{id}$ will provide the required isomorphism of \mathcal{G}_0 -modules. Consider the Lie supergroup

$$\mathcal{G}^3 = (\mathcal{G}_0, \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_1, \mathcal{F}_{\mathcal{G}_0}))$$

from Section 3.4. It follows from (4) that

$$H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}^3}) = \text{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_1, \mathbb{C}) = (\bigwedge \mathfrak{g}_1)^*.$$

Note that the action of \mathcal{G}_0 on $(\bigwedge \mathfrak{g}_1)^*$ by right translations in \mathcal{G}^3 , denoted by $(r'_g)^*$, coincides with the standard action of \mathcal{G}_0 on $(\bigwedge \mathfrak{g}_1)^*$. Indeed, let us take

$$f \in H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})^{\mathcal{G}_0} = (\bigwedge \mathfrak{g}_1)^*.$$

By (11), we have:

$$(r'_g)^*(f)(X)(e) = (r'_g)^*(f)(X)(h) = f(\text{Ad}(g^{-1})X)(hg) = f(\text{Ad}(g^{-1})X)(e).$$

Here $g, h \in \mathcal{G}_0$, $X \in \bigwedge \mathfrak{g}_{\bar{1}}$ and $e \in \mathcal{G}_0$ is the identity. It remains to note that by Lemma 3, the map $\Phi_{\mathfrak{g}}$ induces the equivariant isomorphism between the superspaces of left invariants $H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}})$ and $(\bigwedge \mathfrak{g}_{\bar{1}})^*$.

b. We need to show that in the vector space

$$(\bigwedge (\mathfrak{g}_{\bar{1}}^* \otimes \mathfrak{g})^{\mathcal{G}_0} = (\bigwedge (\mathfrak{g}_{\bar{1}}^* \otimes \mathfrak{g}_0)^{\mathcal{G}_0} \oplus (\bigwedge (\mathfrak{g}_{\bar{1}}^* \otimes \mathfrak{g}_{\bar{1}})^{\mathcal{G}_0}$$

there exists points corresponding to split grading operators. This space always possesses a \mathcal{G}_0 -invariant, precisely, the identity operator $\text{id} \in \mathfrak{g}_{\bar{1}}^* \otimes \mathfrak{g}_{\bar{1}}$. The pre-image of $\beta^{-1}(\text{id}) \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}$ has the form $\sum \varepsilon^i X_i$ for some choice of local coordinates such that $X_i(\varepsilon^j)(e) = \delta_i^j$, see Example 2. We have seen that such vector fields correspond to \mathcal{G}_0 -left invariant split grading operators on \mathcal{G} . \square

Denote by $\mathcal{T}_{\mathcal{G}}$ the tangent sheaf of a Lie group \mathcal{G} and by \bar{v} is the image of v by the natural mapping

$$H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g} \rightarrow H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h}.$$

The result of our study is:

Theorem 1. *The following conditions are equivalent:*

- a.** *A homogeneous supermanifold $\mathcal{M} = \mathcal{G}/\mathcal{H}$ admits a \mathcal{G}_0 -left invariant split grading that is induced by a grading of $\mathcal{O}_{\mathcal{G}}$ and the inclusion $\mathcal{O}_{\mathcal{M}} \subset (\pi_0)_*(\mathcal{O}_{\mathcal{G}})$.*
- b.** *There exists a \mathcal{G}_0 -left invariant vector field $\chi \in H^0(\mathcal{G}_0, (\mathcal{T}_{\mathcal{G}})_{(2)\bar{0}})$ such that*

$$\bar{\chi} \in \left(\bigwedge (\mathfrak{g}_{\bar{1}})^* \otimes \mathfrak{g}/\mathfrak{h} \right)^{\mathcal{H}_0}, \quad (20)$$

and such that for $w = \beta^{-1}(\text{id}) + \chi$, where $\beta^{-1}(\text{id}) = \sum \varepsilon^i X_i$ is from the proof of Proposition 5.b, we have

$$[Y, w] \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{h}, \quad Y \in \mathfrak{h}_{\bar{1}}. \quad (21)$$

6 An application

As above let \mathcal{G} be a Lie supergroup and \mathcal{H} be a Lie subsupergroup, \mathfrak{g} and \mathfrak{h} be the Lie superalgebras of \mathcal{G} and \mathcal{H} , respectively, and $\mathcal{M} := \mathcal{G}/\mathcal{H}$. Consider the map

$$\rho : \mathfrak{g}_0 \rightarrow H^0(\text{pt}, \mathcal{T}_{\mathcal{G}_0 \setminus \mathcal{G}})$$

induced by the action of \mathcal{G}_0 on \mathcal{M} . (Here $\mathcal{T}_{\mathcal{G}_0 \setminus \mathcal{G}}$ is the sheaf of vector fields on $\mathcal{G}_0 \setminus \mathcal{G}$.) Let us describe its kernel. For $X \in \mathfrak{g}_0$ and $f \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}})$, we have:

$$\begin{aligned} X(f)(Y)(e) &= \left. \frac{d}{dt} \right|_{t=0} f(\text{Ad}(\exp(-tX))Y)(\exp(tX)) = \\ &= \left. \frac{d}{dt} \right|_{t=0} f(\text{Ad}(\exp(-tX))Y)(e), \end{aligned}$$

where $Y = Y_1 \cdots Y_r$, $Y_i \in \mathfrak{g}_1$ and t is an even parameter. A vector field X is in $\text{Ker } \rho$ if and only if $X(f)(Y)(e) = 0$ for all f and Y . Hence,

$$\text{Ker } \rho = \text{Ker}(\text{ad}|_{\mathfrak{g}_1}),$$

where ad is the adjoint representation of \mathfrak{g}_0 in \mathfrak{g} .

Furthermore, denote

$$\begin{aligned} A &:= \text{Ker}(\mathcal{G}_0 \ni g \mapsto \bar{l}_g : \mathcal{G}/\mathcal{H} \rightarrow \mathcal{G}/\mathcal{H}); \\ \mathfrak{a} &:= \text{Ker}(\mathfrak{g} \ni X \mapsto H^0(\mathcal{G}_0/\mathcal{H}_0, \mathcal{T}_{\mathcal{G}/\mathcal{H}})). \end{aligned}$$

Here \bar{l}_g is the automorphism of \mathcal{G}/\mathcal{H} induced by the left translation l_g . The pair (A, \mathfrak{a}) is a super Harish-Chandra pair. An action of \mathcal{G} on \mathcal{M} is called *effective* if the corresponding to (A, \mathfrak{a}) Lie supergroup is trivial. As in the case of Lie groups any action of a Lie supergroup can be factored to be effective.

Theorem 2. *Assume that the action of \mathcal{G} on \mathcal{M} is effective. If*

$$[\mathfrak{g}_1, \mathfrak{h}_1] \subset \mathfrak{h}_0 \cap \text{Ker}(\text{ad}|_{\mathfrak{g}_1}),$$

then \mathcal{M} is split.

Proof. Let us show that in this case the vector field $w = \sum \varepsilon^i X_i + 0 = \sum \varepsilon^i X_i$ from Proposition 5.b is a (left invariant) split grading operator on \mathcal{M} using Theorem 1.

The condition (20) is satisfied trivially, because $\chi = 0$. Let us check the condition (21). We have:

$$[Y, v] = \sum Y(\varepsilon^i)X_i - \sum \varepsilon^i [Y, X_i].$$

Since $[\mathfrak{g}_1, \mathfrak{h}_1] \subset \mathfrak{h}_0$, we get

$$\sum \varepsilon^i [Y, X_i] \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{h}.$$

Hence, we have to show that

$$\sum Y(\varepsilon^i)X_i \in H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{h}.$$

Assume that X_1, \dots, X_k is a basis of $\mathfrak{h}_{\bar{1}}$, $X_1, \dots, X_k, X_{k+1}, \dots, X_m$ is a basis of $\mathfrak{g}_{\bar{1}}$ and (ε^i) is the system of global odd \mathcal{G}_0 -left invariant coordinates corresponding to this basis such that $\sum \varepsilon^i X_i$ is as in Proposition 5.b. In particular, $\varepsilon^i(\gamma_{\mathfrak{g}}(X_j)) = \delta_j^i$, because $\sum(\varepsilon^i \circ \gamma_{\mathfrak{g}}) \otimes X_i$ is the identity operator in $\mathfrak{g}_{\bar{1}}^* \otimes \mathfrak{g}_{\bar{1}}$.

Let us take $Z \in \text{Ker } \rho$. Clearly, $Z(\varepsilon^i) = 0$ and $X_j(\varepsilon^i)$ is again a \mathcal{G}_0 -left invariant function on \mathcal{G} . By (8), we also have:

$$\varepsilon^i(X_{i_1} \cdots Z \cdots X_{i_k}) = 0.$$

Furthermore, by definition of ε^i , we get that $\varepsilon^i \circ \gamma_{\mathfrak{g}} \in \mathfrak{g}_{\bar{1}}^*$. Hence,

$$\varepsilon^i(\gamma_{\mathfrak{g}}(X_{i_1} \wedge \cdots \wedge X_{i_k})) = 0,$$

if $k > 1$. Summing up all these observations we see that

$$\varepsilon^i(\gamma_{\mathfrak{g}}(X) \cdot Y) = \varepsilon^i(\gamma_{\mathfrak{g}}(X \wedge Y)) + 0,$$

where $Y \in \mathfrak{h}$ and $X \in \bigwedge \mathfrak{g}_{\bar{1}}$. Now we can conclude that

$$\sum Y(\varepsilon^i)X_i = -Y \in \mathfrak{h} \subset H^0(\text{pt}, \mathcal{O}_{\mathcal{G}_0 \setminus \mathcal{G}}) \otimes \mathfrak{h}.$$

The proof is complete. \square

Example 3. Consider the super-grassmannian $\mathbf{Gr}_{m|n,k|l}$. It is a $\text{GL}_{m|n}$ -homogeneous space, see [9] for more details. Hence, $\mathbf{Gr}_{m|n,k|l} \simeq \text{GL}_{m|n}/\mathcal{H}$ for a certain \mathcal{H} . (See, for example, [15].) In the case $k = 0$ or $k = m$, the following holds $[(\mathfrak{gl}_{m|n})_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = 0$. Therefore, by Theorem 2, the super-grassmannian is split.

In [9] it was shown that the super-grassmannian $\text{GL}_{m|n,k|l}$ is not split if and only if $0 < k < m$ and $0 < l < n$. (This fact also follows from results in [6] and [13] about non-projectivity of super-grassmannian.)

Finally, let us recall a result proved in [14]:

Theorem 3. *If a complex homogeneous supermanifold \mathcal{M} is split, then there is a Lie supergroup \mathcal{G} with $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$, where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \text{Lie } \mathcal{G}$, such that \mathcal{G} acts on \mathcal{M} transitively.*

References

- [1] *Berezin F.A., Leites D.A.* Supermanifolds. Soviet Math. Dokl. 16, 1975, Pages 1218-1222.

- [2] *Donagi R., Witten E.* Supermoduli space is not projected. arXiv:1304.7798, 57 Pages, 2013.
- [3] *Green P.* On holomorphic graded manifolds. Proc. Amer. Math. Soc. 85, no. 4, 1982, Pages 587-590.
- [4] *Koszul J.-L.* Connections and splittings of supermanifolds. Differential Geom. Appl. 4, no. 2, 1994, Pages 151-161.
- [5] *Koszul J.-L.* Graded manifolds and graded Lie algebras. Proceeding of the International Meeting on Geometry and Physics (Bologna), Pitagora, 1982, Pages 71-84.
- [6] *LeBrun C., Poon Y. S., Wells R. O.* Projective embeddings of complex supermanifolds. Comm. Math. Phys. 126, no. 3, 1990, Pages 433-452.
- [7] *Leites D.A.* Introduction to the theory of supermanifolds. Russian Math. Surveys 35, 1980, Pages 1-64.
- [8] *Manin Yu.I.* Gauge field theory and complex geometry, Grundlehren der Mathematischen Wissenschaften, V. 289, Springer-Verlag, Berlin, second edition, 1997.
- [9] *Onishchik A.L., Serov A.A.* Holomorphic vector fields on super-Grassmannians. Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math., 8, Amer. Math. Soc., 1992, Providence, RI, Pages 113-129.
- [10] *Onishchik A.L., Serov A.A.* Vector fields and deformations of isotropic super-Grassmannians of maximal type. Lie Groups and Lie Algebras: E.B. Dynkin's Seminar, AMS Transl. Ser. 2. V. 169. Providence: AMS, 1995, Pages 75-90.
- [11] *Onishchik A.L., Serov A.A.* On isotropic super-Grassmannians of maximal type associated with an odd bilinear form. E. Schrödinger Inst. for Math. Physics, Preprint No. 340. Vienna,
- [12] *Onishchik A.L., Vishnyakova E. G.* Locally free sheaves on complex supermanifolds. Transformation groups, Volume 18, Issue 2, 2013, Pages 483-505.
- [13] *Penkov I. B., Skorniyakov I. A.* Projectivity and D-affineness of flag supermanifolds. (Russian) Uspekhi Mat. Nauk 40, no. 1(241), 1985, Pages 211-212.

- [14] *Vishnyakova E.G.* On complex Lie supergroups and split homogeneous supermanifolds. Transformation groups, Vol. 16, Issue 1, 2011, Pages 265 - 285.
- [15] *Vishnyakova E.G.* On holomorphic functions on a compact complex homogeneous supermanifold. Journal of Algebra, Volume 350, Issue 1, 2012, Pages 174-196.
- [16] *Vishnyakova E.G.* On the Structure of Complex Homogeneous Supermanifolds. arXiv:0811.2581, 2008, 12 Pages.

Elizaveta Vishnyakova

Max Planck Institute for Mathematics Bonn and
University of Luxembourg

E-mail address: VishnyakovaE@googlemail.com